Inverse problem

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An **inverse problem** is the task that often occurs in many branches of science and mathematics where the values of some model parameter(s) must be obtained from the observed data.

The inverse problem can be formulated as follows:

Data → Model parameters

The transformation from data to model parameters is a result of the interaction of a *physical system*, e.g., the Earth, the atmosphere, gravity etc. Inverse problems arise for example in geophysics, medical imaging (such as computed axial tomography), remote sensing, ocean acoustic tomography, nondestructive testing, and astronomy.

Inverse problems are typically ill posed, as opposed to the well-posed problems more typical when modeling physical situations where the model parameters or material properties are known. Of the three conditions for a well-posed problem suggested by Jacques Hadamard
(existence, uniqueness, stability of the solution or solutions) the condition of stability is most often violated. In the sense of functional analysis, the inverse problem is represented by a mapping between metric spaces. While inverse problems are often formulated in infinite dimensional spaces, limitations to a finite number of measurements, and the practical consideration of recovering only a finite number of unknown parameters, may lead to the problems being recast in discrete form. In this case the inverse problem will typically be *ill-conditioned*. In these cases, regularization
may be used to introduce mild assumptions on the solution and prevent overfitting. Many instances of regularized inverse problems can be interpreted as special cases of Bayesian inference.

Contents

Linear inverse problems

A linear inverse problem can be described by:

$$d = G(m)$$

where *G* is a linear operator
describing the explicit relationship between data and model parameters, and is a representation of the physical system. In the case of a discrete linear inverse problem describing a linear system, *d* and *m* are vectors, and the

problem can be written as

$$d = Gm$$

where *G* is a matrix.

Examples

One central example of a linear inverse problem is provided by a Fredholm first kind integral equation.

$$d(x) = \int_a^b g(x,y)\, m(y)\, dy$$

For sufficiently smooth *g* the operator defined above is compact on reasonable Banach spaces such as L^p spaces. Even if the mapping is injective its inverse
will not be continuous. (However, by the bounded inverse theorem, if the mapping is bijective, then the inverse will be bounded (ie. continuous).) Thus small errors in the data *d* are greatly amplified in the solution *m*. In this sense the inverse problem of inferring *m* from measured *d* is ill-posed.

To obtain a numerical solution, the integral must be approximated using quadrature, and the data sampled at discrete points. The resulting system of linear equations will be ill-conditioned.

Another example is the inversion of the Radon transform. Here a function (for example of two variables) is deduced from its integrals along all possible lines. This is precisely the problem solved in image reconstruction for X-ray computerized tomography. Although from a theoretical point of view many linear inverse problems are well understood, problems involving the Radon transform and its generalisations still present many theoretical challenges with questions of sufficiency of data still unresolved. Such problems include incomplete data for the x-ray transform in three dimensions and problems involving the generalisation of the x-ray transform to tensor fields.

Non-linear inverse problems

An inherently more difficult family of inverse problems are collectively referred to as non-linear inverse problems.

Non-linear inverse problems have a more complex relationship between data and model, represented by the equation:

$$d = G(m).$$

Here *G*
is a non-linear operator and cannot be separated to represent a linear mapping of the model parameters that form *m* into the data. In such research, the first priority is to understand the structure of the problem and to give a theoretical answer to the three Hadamard questions (so that the problem is solved from the theoretical point of view). It is only later in a study that regularization and interpretation of the solution's (or solutions', depending upon conditions of uniqueness) dependence upon parameters and data/measurements (probabilistic ones or others) can be done. Hence the corresponding following sections do not really apply to these problems. Whereas linear inverse problems were completely solved from the theoretical point of view at the end of the nineteenth century, only one class of nonlinear inverse problems was so before 1970, that of inverse spectral and (one space dimension) inverse scattering problems, after the seminal work of the Russian mathematical school (Krein, Gelfand, Levitan, Marchenko). A large review of the results has been given by Chadan and Sabatier in their book "Inverse Problems of Quantum Scattering Theory" (two editions in English, one in Russian).

In this kind of problems, data are properties of the spectrum of a linear operator which describe the scattering. The spectrum is made of eigenvalues and eigenfunctions, forming together the "discrete spectrum", and generalizations, called the continuous spectrum. The very remarkable physical point is that scattering experiments give information only on the continuous spectrum, and that knowing its full spectrum is both necessary and sufficient in recovering the scattering operator. Hence we have invisible parameters, much more interesting than the null space which has a similar property in linear inverse problems. In addition, there are physical motions in which the spectrum of such an operator is conserved as a consequence of such motion. This phenomenon is governed by special nonlinear partial differential evolution equations, for example the Korteweg–de Vries equation. If the spectrum of the operator is reduced to one single eigenvalue, its corresponding motion is that of a single bump that propagates at constant velocity and without deformation, a solitary wave called "soliton".

A perfect signal and its generalizations for the Korteweg–de Vries equation or other integrable nonlinear partial differential equations are of great interest, with many possible applications. This area has been studied as a branch of mathematical physics since the 1970s. Nonlinear inverse problems are also currently studied in many fields of applied science (acoustics, mechanics, quantum mechanics, electromagnetic scattering - in particular radar soundings, seismic soundings and nearly all imaging modalities).

Inverse Problems Societies

- Inverse Problems International Association (http://www.inverse-problems.net/)
- Finnish Inverse Problems Society (http://venda.uku.fi/research/FIPS)

External links

- Inverse Problems Network (http://www.mth.msu.edu/ipnet/)
- Inverse Problems page at the University of Alabama (http://www.me.ua.edu/inverse/)
- Another Inverse Problems web site (http://www.inverse-problems.com/)
- Albert Tarantola's website, including a free PDF version of his Inverse Problem Theory book, and some on-line articles on Inverse Problems (http://www.ipgp.jussieu.fr/~tarantola/)
- Andy Ganse's Geophysical Inverse Theory Resources Page (http://staff.washington.edu/aganse/invresources/index.html)
- Finnish Centre of Excellence in Inverse Problems Research (http://math.tkk.fi/inverse-coe)

Academic journals

There are four main academic journals covering inverse problems in general.

- Inverse Problems (http://www.iop.org/EJ/journal/IP)
- Journal of Inverse and Ill-posed Problems (http://www.vsppub.com/journals/jn-JouInvIllPosPro.html)
- Inverse Problems in Science and Engineering (http://www.tandf.co.uk/journals/titles/17415977.asp)
- Inverse Problems and Imaging (http://aimsciences.org/journals/ipi/ipi_online.jsp)

In addition there are many journals on medical imaging, geophysics, non-destructive testing etc that are dominated by inverse problems in those areas.

References

- Chadan, Khosrow & Sabatier, Pierre Célestin (1977). *Inverse Problems in Quantum Scattering Theory*. Springer-Verlag. ISBN 0387080929

- Aster, Richard [et al.] (2004). *Parameter Estimation and Inverse Problems*, Elsevier. ISBN 978-0-12-065604-2; ISBN10 0-12-065604-3

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